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APPROACHED BOLZA TYPE PROBLEMS IN DISCRETE TIME

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ABSTRACT. Our aim is to give approximate optimality conditions for Bolza type problem in discrete time with finite dimensional in non-smooth analysis.We have just applied the subdifferential calculus of Mordukhovich(see,e.g., [1],[2],[3]) on the one hand and on the other hand, the Ekeland's variational principle (see,e.g., [20],[21],[22]).

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1. INTRODUCTION

The differential of a convex function generalizes the notion of derivative and provided its first examples in the theory of maximal monotone operators, so successful for partial differential equations and integral equations. Another motivation also is an adjunct of working without convexity, called nonsmooth analysis. This paper is based on the limiting Frechet sub-differential which is introduced by Mordukhovich for several reasons: in particular because this sub-differential is "smaller" than of Clarke(see[2], [29], [1]), it contains fewer errors and it is interesting for the riches their calculations. One objective of the optimization is to establish necessary and sufficient optimality conditions if possible. The existence of the optimum solution is ensured for example by the compactness of the domain and the semi-continuity of the objective. And if one of these conditions is not satisfied the problem may not have exact solutions, but if the objective is bounded from below, the infimum exists without being hit. It is for this reason Ekeland thought to the notion of the approximate solution. And he was the first one in 1972, whose gave the approximate necessary optimality conditions (see [20, 21, 22, 23]). So, our paper carries on the one hand on the Ekeland's variational principle, which has been a very importance in nonlinear analysis, and enjoyed a great variation of applications ranging from geometry Banach spaces, in the optimization theory and subdifferential generalized calculation to calculate the variation. And secondly on the non-smooth analysis in order to generalized approximate optimality conditions for Bolza type problems with constraints in discrete time(see, e.g., [31],[29]).

First of all, let us recall the form of general Bolza type problem in the Variations Calculus in Non-smooth Analysis which can be formulated as the minimization of the functional

$$I(x) = l(x(t_0), x(t_1)) + \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))\, dt, \tag{1}$$

over the space $W_n^{1,1}([t_0, t_1]) := W^{1,1}([t_0, t_1], \mathbb{R}^n)$ of all absolutely continuous arcs $x : [t_0, t_1] \to \mathbb{R}^n$ whose derivatives $\dot{x}(\cdot)$ (defined almost everywhere) belong to $L_n^1([t_0, t_1])$ and such that for all $t \in [t_0, t_1]$,

$$x(t) = x(0) + \int_{t_0}^{t_1} \dot{x}(s) ds.$$

Since the functions l and L are assumed to take their values in $\mathbb{R} \cup \{+\infty\}$. This formulation incorporates the equalities and inequalities constraints relative to the initial end point pair $(x(t_0), x(t_1))$ and possibly nonsmooth set constraints and set-valued constraints.

In the corresponding discrete time problem, one considers in place of an arc $x : [t_0, t_1] \to \mathbb{R}^n$ a vector $x = (x_0, x_1, \cdots, x_T) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n = (\mathbb{R}^n)^{T+1}$ i.e, a discrete arc, and in place of $\dot{x} = \frac{dx}{dt}$ the difference $\Delta x_t = x_t - x_{t-1}$ for $t = 1, \cdots, T$. The associated problem $(\mathcal{P}(l, L))$ takes then the form: Minimize over all $x = (x_0, x_1, \cdots, x_T) \in (\mathbb{R}^n)^{T+1}$ the function

$$J(x) := l(x_0, x_1) + \sum_{t=1}^{T} L_t(x_{t-1}, \Delta x_t), \tag{2}$$

where l and L_t for all $t = 1, \cdots, T$ are functions from $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R} \cup \{+\infty\}$ which are *proper*, that is, none of which is identically $+\infty$. Throughout, we assume that these functions are *lower semicontinuous* (lsc, for short) or locally Lipschitzian. Then J, too, is lsc with values in $\mathbb{R} \cup \{+\infty\}$. As for the Bolza type problem above in variations calculus, it is important to observe the fact that in $(\mathcal{P}(l, L))$ the constraints are implicit in the inequality $J(x) < \infty$, because only vectors x satisfying $J(x) < +\infty$ are of interest in the minimization.

The discrete Bolza type problem $(\mathcal{P}(l, L))$ has been introduced and largely studied in the *convex* setting by Rockafellar and Wets [29]. Results concerning the discrete Bolza type problem in the form $(\mathcal{P}_{C,F}(l, L))$, with l and L locally Lipschitzian, have been also provided in Mordukhovich [2][Theorem 6.17], see [1]. It is also worth mentioning that, among domains of its own interest, the problem $(\mathcal{P}_{C,F}(l, L))$ contains as particular case the modelization of various economic dynamics see,[2]. In the present paper we focus our attention to the discrete problem without any convexity assumption.

Throughout, we assume that J is proper, that is, there exists some $z \in (\mathbb{R}^n)^{T+1}$ such that $l(z_0, z_T) < +\infty$ and $L_t(z_{t-1}, \Delta z_t) < +\infty$ for all $t = 1, \cdots, T$. Letting

$$C := \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^n \mid l(u, v) < \infty\}, \tag{3}$$

(that is, C is the effective domain $\operatorname{dom} l$ of l) and

$$F_t(u) := \{v \in \mathbb{R}^n \mid L_t(u, v) < \infty\}, \tag{4}$$

it is emphasized in [29] that, without loss of generality, one can restrict attention in $(\mathcal{P}(l, L))$ to minimizing $J(x)$ over the set of all $x \in (\mathbb{R}^n)^{T+1}$

which satisfy

(5) $$(x_0, x_T) \in C \text{ and } \Delta x_t \in F_t(x_{t-1}), \forall\, t = 1, \cdots, T.$$

Implicit in the dynamical constraint $\Delta x_t \in F_t(x_{t-1})$ is the state constraint $x_{t-1} \in Z_t$ for $t = 1, \cdots, T$, where

$$Z_t = \{z \in \mathbb{R}^n | F_t(z) \neq \emptyset\}.$$

Conversely, given finite valued functions l and L, set constraint $C \subset \mathbb{R}^n \times \mathbb{R}^n$ and set-valued mapping constraints $F_t : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, it is also of interest to study the minimization problem $(\mathcal{P}_{C,F}(l, L))$ consisting in minimizing the function $J(\cdot)$ above over all the vectors $x \in (\mathbb{R}^n)^{T+1}$ satisfying the constraints in 5. At a first step, this problem may be translated in the form of problem $(P(l, L))$ by putting on the one hand $\tilde{l}(u, v) = l(u, v)$ if $(u, v) \in C$ and $\tilde{l}(u, v) = +\infty$ otherwise and one the other hand $\tilde{L}_t(u, v) = L_t(u, v)$ if $v \in F_t(u)$ and $\tilde{L}_t(u, v) = +\infty$ otherwise.

2. Definitions and preliminaries

In the next section, although our approached optimality conditions could be given with the use of *many types of subdifferentials*, we will limit ourselves to state and establish them with the basic limiting subdifferential. Recall first that for a proper lsc function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $u \in \operatorname{dom} f$ the Fréchet subdifferential $\hat{\partial} f(u)$ is defined by the fact that a vector $v \in \hat{\partial} f(u)$ when for any positive number ε there exists some positive number η such that one has

$$\langle v, u' - u\rangle \leq \varepsilon \|u' - u\| \text{ for all } u' \in \mathbb{B}(u, \eta),$$

where $B(u, \eta)$ denotes the open ball with radius η centered at the point u. One puts in general $\hat{\partial} f(u) = \emptyset$ when $f(u)$ is not finite, see, [2]. When f is the indicator function δ_S of a closed subset $S \subset \mathbb{R}^n$, that is, $\delta_S(u) = 0$ if $u \in S$ and $\delta_S(u) = +\infty$ otherwise, its Frechet subdifferential at a point $u \in S$ is a cone. It is generally called the Frechet normal cone to S at u and one denotes either $\hat{N}_S(u)$ or $\hat{N}(S, u)$. Since the Frechet subdifferential enjoys only fuzzy calculus rules (see, e.g., [2] for more details), one considers a limiting process of such subdifferentials yielding to the so-called limiting subdifferential. A vector v is in the limiting subdifferential $\partial f(u)$ at a point $u \in \operatorname{dom} f$ when there exists a sequence $(u_k, f(u_k))$ converging to $(u, f(u))$ and vectors $v_k \in \hat{\partial} f(u_k)$ with $v_k \to v$. As above, one sets $\partial f(u) = \emptyset$ if $u \notin \operatorname{dom} f$. The set $\partial f(u)$ is nonconvex in general but it enjoys full point based calculus rules. For example, if $g : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function one has (see [2, 29]) the inclusion

(6) $$\partial(f + g)(u) \subset \partial f(u) + \partial g(u),$$

where the addition in the second member is taken in the usual Minkowski sense, that is,

$$\partial f(u) + \partial g(u) := \{v + v' \,|\, v \in \partial f(u),\, v' \in \partial g(u)\}.$$

The inclusion (6) can be also obtained under a much weaker condition than the local Lipschitz property of one of the functions f and g. To see that, let us recall the concept of singular limiting subdifferential. Modifying slightly the definition above, we say that a vector v belongs to the singular limiting subdifferential $\partial^\infty f(u)$ at a point $u \in \operatorname{dom} f$ when there exists a sequence $(u_k, f(u_k))$ converging to $(u, f(u))$, positive numbers $\lambda_k \downarrow 0$ and vectors $v_k \in \hat{\partial} f(u_k)$ such that $\lambda_k v_k \to v$. So, if for two lsc functions f, g the qualification condition $\partial^\infty f(u) \cap (-\partial^\infty g(u)) = \{0\}$ holds, then one has see, e.g., [2, 29] $\partial(f+g)(u) \subset \partial f(u) + \partial g(u)$. This qualification condition can be translated see, e.g., [2, 29] in the case of any finite number of lsc functions: for a finite number of lsc functions f_i, $i = 0, 1, \cdots, m$, and for $u \in \cap_{i=0}^m \operatorname{dom} f_i$ one has

$$\partial(\sum_{i=0}^m f_i)(u) \subset \sum_{i=0}^m \partial f_i(u), \tag{7}$$

whenever for any $y_i \in \partial^\infty f_i(u)$ with $\sum_{i=0}^m y_i = 0$ one necessarily has $y_0 = y_1 = \cdots = y_m = 0$. The inclusion (6) is a particular case of (7) since

$$\partial^\infty g(u) = \{0\} \text{ whenever } f \text{ Lipschitz near } u. \tag{8}$$

The same qualification condition above also gives (see, e.g., [2, 29])

$$\partial^\infty(\sum_{i=0}^m f_i)(u) \subset \sum_{i=0}^m \partial^\infty f_i(u). \tag{9}$$

Concerning the composition operation, we will recall the result with the composition with a linear mapping. If $A : \mathbb{R}^m \to \mathbb{R}^n$ is a linear *surjective* mapping, then see [2, 29]

$$\partial(f \circ A)(u) \subset A^* \partial f(Au) \text{ and } \partial^\infty(f \circ A)(u) \subset A^* \partial^\infty f(Au), \tag{10}$$

where A^* denotes the adjoint of A and $A^*\partial f(Au) := \{A^* v \mid v \in \partial f(Au)\}$. As for the Frechet normal cone (see above), the limiting normal cone to a closed subset S at $u \in S$ is defined through its indicator function by $N_S(u) := \partial \delta_S(u)$. Sometimes one write $N(S, u)$ in place of $N_S(u)$. The connexion with the singular subdifferential is provided by the equalities

$$\partial^\infty \delta_S(u) = \partial \delta_S(u) = N(S, u).$$

Of course, when the point u is a minimum point for the function f one has both $0 \in \hat{\partial} f(u)$ and $0 \in \partial f(u)$, the first inclusion being obvious under the minimum point assumption and the second one being a consequence of the fact that one always has $\hat{\partial} f \subset \partial f$. Further, when f is convex, the Frechet subdifferential and the limiting subdifferential coincide with the usual Fenchel subdifferential of Convex Analysis. In the next section, we will just say subdifferential of f and normal cone to S in place of limiting subdifferential of f and limiting normal cone to S.

We based in our research on several articles that are cited in the bibliography of this paper, so let us recall the two theorems which we based on, for our extenton (see, [31],[21]).

Theorem 2.1. *Let $\bar{x} \in (\mathbb{R}^n)^{T+1}$ be a solution of problem $(P(l, L))$.*

Assume that l and L_t are proper and lsc for all $t = 1, \cdots, T$ and that the following qualification condition $Q(\bar{x})$ holds:

$$\begin{cases} \text{the only vector } y = (y_0, \cdots, y_T) \in (\mathbb{R}^n)^{T+1} \text{for which} \\ (y_0, -y_T) \in \partial_\infty l(\bar{x}_0, \bar{x}_T) \text{ and } (\Delta y_t, y_t) \in \partial_\infty L_t(\bar{x}_{t-1}, \Delta\bar{x}_t), \forall t = 1, \cdots, T \\ \text{is the zero vector in } (\mathbb{R}^n)^{T+1}. \end{cases}$$

There exists some vector $p = (p_0, \cdots, p_T) \in (\mathbb{R}^n)^{T+1}$ such that:
a) $(p_0, -p_T) \in \partial l(\bar{x}_0, \bar{x}_T)$.
b) $(\Delta p_t, p_t) \in \partial L_t(\bar{x}_{t-1}, \Delta\bar{x}_t)$ for all $t = 1, \cdots, T$.

Theorem 2.2. *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ a proper function, lsc. and bounded from below on $\mathbb{R}^n$. For each $\varepsilon > 0$ and each $u \in \mathbb{R}^n$ such that:*

$$f(u) \leq \inf_{\mathbb{R}^n} f + \varepsilon.$$

For each $\lambda > 0$, there exists $v \in \mathbb{R}^n$ such that:

$$f(v) \leq f(u), \tag{11}$$

$$\|v - u\| \leq \lambda, \tag{12}$$

$$f(v) < f(x) + \frac{\varepsilon}{\lambda} \|x - v\|, \forall x \in \mathbb{R}^n, \forall x \neq v. \tag{13}$$

Consider the following minimization problem with constraints:

$$(P) \begin{cases} \min f(x) \\ x \in S. \end{cases}$$

Where $f : \mathbb{R}^n \to \mathbb{R}$ is *l.s.c.*, S a nonempty closed subset of $\mathbb{R}^n$, and f is bounded from below on S. For $\varepsilon > 0$ fixed, we say that $u \in S$ is an approximate solution of (P), where $f(u) \leq \inf_S f + \varepsilon$. The approximate necessary optimality condition, adapted to the present context, gives:

Corollary 2.3. *Let u an ε-minimizer of f on S. Then, for each $\lambda > 0$, there exists $v \in S$ such that:*

$$f(v) \leq f(u),$$
$$\|v - u\| \leq \lambda,$$
$$f(v) < f(x) + \frac{\varepsilon}{\lambda} \|x - v\|, \forall x \in S, x \neq v.$$

3. Approached optimality conditions

The following result states the basic theorem of the paper. Here the functions l and L_t are neither convex nor locally Lipschitzian.

Theorem 3.1. *Assume that l and L_t are proper, lsc and bounded from below for all $t = 1, \cdots, T$. So for each $\varepsilon > 0$ and each $u \in \mathbb{R}^n$ such that:*

$$J(u) \leq \inf_{\mathbb{R}^n} J + \varepsilon.$$

There exists $x_\varepsilon \in \mathbb{R}^n$, an approximate solution of $(\mathcal{P}(l, L))$ and that the following qualification condition $Q(x_\varepsilon)$ holds:

$$\begin{cases} \text{the only vector } y = (y_0, \cdots, y_T) \in (\mathbb{R}^n)^{T+1} \text{for which} \\ (y_0, -y_T) \in \partial_\infty l(x_0^\varepsilon, x_T^\varepsilon) \text{ and } (\Delta y_t, y_t) \in \partial_\infty L_t(x_{t-1}^\varepsilon, \Delta x_t^\varepsilon), \forall t = 1, \cdots, T \\ \text{is the zero vector in } (\mathbb{R}^n)^{T+1}. \end{cases}$$

For each $\lambda > 0$, there exists some vector $p_\varepsilon = (p_0^\varepsilon, \cdots, p_T^\varepsilon) \in (\mathbb{R}^n)^{T+1}$ such that:
a) $(p_0^\varepsilon, -p_T^\varepsilon) \in \partial l(x_0^\varepsilon, x_T^\varepsilon) + \frac{\varepsilon}{\lambda}\mathbb{B}^*_{(\mathbb{R}^n)^2}$.
b) $(\Delta p_t^\varepsilon, p_t^\varepsilon) \in \partial L_t(x_{t-1}^\varepsilon, \Delta x_t^\varepsilon) + \frac{\varepsilon}{\lambda}\mathbb{B}^*_{(\mathbb{R}^n)^2}$ for all $t = 1, \cdots, T$.

Proof. The function $J(\cdot)$ are proper, lsc and bounded from below in $(\mathbb{R}^n)^{T+1}$, because l and L_t are proper, lsc and bounded from below for all $\forall t = 1, \cdots, T$. we can apply the Ekeland's variational principle (see Theorem2.2 to the function J. According to this result, $\forall \varepsilon, \lambda > 0$, there exist an approximate solution x_ε,which satisfies

$$J(x_\varepsilon) < J(x) + \frac{\varepsilon}{\lambda}\|x - x_\varepsilon\|, \forall x \neq x_\varepsilon.$$

Now, we consider

$$J_\varepsilon(x) = J(x) + \frac{\varepsilon}{\lambda}\|x - x_\varepsilon\|$$

Therefore, $J_\varepsilon(x_\varepsilon) < J_\varepsilon(x), \forall x \neq x_\varepsilon$.So, x_ε is a minimum of the function $J_\varepsilon(\cdot)$. Knowing that

$$\begin{aligned} J_\varepsilon(x) &= J(x) + \tfrac{\varepsilon}{\lambda}\|x - x_\varepsilon\| \\ &= l(x_0, x_T) + \sum_{t=1}^{T} L_t(x_{t-1}, \triangle x_t) + \varepsilon\|x - x_\varepsilon\| \\ &= l(x_0, x_T) + \sum_{t=1}^{T} L_t(x_{t-1}, \triangle x_t) + \tfrac{\varepsilon}{\lambda}\sum_{t=0}^{T}\|x_t - x_t^\varepsilon\| \\ &= l(x_0, x_T) + \sum_{t=1}^{T} L_t(x_{t-1}, \triangle x_t) + \tfrac{\varepsilon}{\lambda}\|x_0 - x_0^\varepsilon\| + \tfrac{\varepsilon}{\lambda}\sum_{t=1}^{T}\|x_t - x_t^\varepsilon\| \\ &= \left(l(x_0, x_T) + \tfrac{\varepsilon}{\lambda}\|x_0 - x_0^\varepsilon\|\right) + \sum_{t=1}^{T}\left(L_t(x_{t-1}, \triangle x_t) + \tfrac{\varepsilon}{\lambda}\|x_t - x_t^\varepsilon\|\right). \end{aligned}$$

Next consider the functions l_ε and $L_{\varepsilon,t}$ defined by:

$$\begin{aligned} l_\varepsilon(x_0, x_T) &= l(x_0, x_T) + \tfrac{\varepsilon}{\lambda}\|x_0 - x_0^\varepsilon\|_{\mathbb{R}^n} \\ L_{\varepsilon,t}(x_{t-1}, \triangle x_t) &= L_t(x_{t-1}, \triangle x_t) + \tfrac{\varepsilon}{\lambda}\|x_t - x_t^\varepsilon\|_{\mathbb{R}^n}. \end{aligned}$$

Then we have: $J_\varepsilon(x) = l_\varepsilon(x_0, x_T) + \sum_{t=1}^{T} L_{\varepsilon,t}(x_{t-1}, \triangle x_t)$, and consider the problem noted $(P_{\varepsilon,\det})$:

Minimize over all $x = (x_0, x_1, \cdots, x_T) \in (\mathbb{R}^n)^{T+1}$ the function

$$J_\varepsilon(x) = l_\varepsilon(x_0, x_T) + \sum_{t=1}^{T} L_{\varepsilon,t}(x_{t-1}, \triangle x_t), \tag{14}$$

and consequently x_ε is a solution of the problem $(P_{\varepsilon,\det})$ and $J_\varepsilon(\cdot)$ is proper function, $l.s.c$ and bounded from below on $\mathbb{R}^n$. and that the problem $(P_{\varepsilon,\det})$ satisfies the qualification condition at x_ε, because $\partial_\infty \|.\| = \{0\}$. Now we can apply the basic theorem of Sahraoui [31] on $(P_{\varepsilon,\det})$, which gives us the existence of a vector $p_\varepsilon = (p_0^\varepsilon, \cdots, p_T^\varepsilon) \in (\mathbb{R}^n)^{T+1}$ such that:

a) $(p_0^\varepsilon, -p_T^\varepsilon) \in \partial l_\varepsilon(x_0^\varepsilon, x_T^\varepsilon)$

b) $(\triangle p_t^\varepsilon, p_t^\varepsilon) \in \partial L_{t,\varepsilon}(x_{t-1}^\varepsilon, \triangle x_t^\varepsilon), \forall t = 1, \cdots, T,$

we can easily write:

$$\begin{aligned} &l_\varepsilon(x_0, x_T) = \left(l + \delta_{x_0^\varepsilon} \circ F_0\right)(x_0, x_T), \\ &L_{t,\varepsilon}(x_{t-1}, x_t) = \left(L_{t,\varepsilon} + \delta_{x_t^\varepsilon} \circ F\right)(x_{t-1}, \triangle x_t), \forall\, t = 1, \cdots, T. \end{aligned}$$

knowing that

$$\begin{aligned} &F_0(a,b) = a, F_0^*(a) = (a,0), \forall (a,b) \in (\mathbb{R}^n)^2, \\ &F(a,b) = a+b, F^*(a) = (a,a), \forall (a,b) \in (\mathbb{R}^n)^2, \end{aligned}$$

and $\delta_{x_t^\varepsilon}(y) = \frac{\varepsilon}{\lambda}\|y - x_t^\varepsilon\|, \forall t = 1, \cdots, T, \forall y \in \mathbb{R}^n$. By applying the subdifferential calculus, see [2] one will have,

$$\begin{aligned} \partial l_\varepsilon(x_0^\varepsilon, x_T^\varepsilon) &= \partial\left(l + \delta_{x_t^\varepsilon} \circ F_0\right)(x_0^\varepsilon, x_T^\varepsilon) \\ &\subset \partial l(x_0^\varepsilon, x_T^\varepsilon) + \partial(\delta_{0,x_\varepsilon} \circ F_0)(x_0^\varepsilon, x_T^\varepsilon) \\ &\subset \partial l(x_0^\varepsilon, x_T^\varepsilon) + F_0^*\left(\partial \delta_{x_0^\varepsilon}(F_0(x_0^\varepsilon, x_T^\varepsilon))\right) \\ &\subset \partial l(x_0^\varepsilon, x_T^\varepsilon) + F_0^*\left(\partial \delta_{x_0^\varepsilon}(x_0^\varepsilon)\right) \\ &\subset \partial l(x_0^\varepsilon, x_T^\varepsilon) + \tfrac{\varepsilon}{\lambda}\mathbb{B}_{(\mathbb{R}^n)^2}, \end{aligned}$$

and also that

$$\begin{aligned} \partial L_{t,\varepsilon}(x_{t-1}^\varepsilon, \triangle x_t^\varepsilon) &= \partial\left(L_{t,\varepsilon} + \delta_{x_t^\varepsilon} \circ F\right)(x_{t-1}^\varepsilon, \triangle x_t^\varepsilon) \\ &\subset \partial L_{t,\varepsilon}(x_{t-1}^\varepsilon, \triangle x_t^\varepsilon) + \partial\left(\delta_{x_t^\varepsilon} \circ F\right)(x_{t-1}^\varepsilon, \triangle x_t^\varepsilon) \\ &\subset \partial L_{t,\varepsilon}(x_{t-1}^\varepsilon, \triangle x_t^\varepsilon) + F^*\left(\partial \delta_{x_t^\varepsilon}\left(F(x_{t-1}^\varepsilon, \triangle x_t^\varepsilon)\right)\right) \\ &\subset \partial L_{t,\varepsilon}(x_{t-1}^\varepsilon, \triangle x_t^\varepsilon) + F^*\left(\partial \delta_{x_t^\varepsilon}\left(x_{t-1}^\varepsilon + \triangle x_t^\varepsilon\right)\right) \\ &\subset \partial L_t(x_{t-1}^\varepsilon, \triangle x_t^\varepsilon) + \tfrac{\varepsilon}{\lambda}\mathbb{B}_{(\mathbb{R}^n)^2}, \end{aligned}$$

which gives

$$\begin{aligned} (p_0^\varepsilon, -p_T^\varepsilon) &\in \partial l(x_0^\varepsilon, x_T^\varepsilon) + \frac{\varepsilon}{\lambda}\mathbb{B}_{(\mathbb{R}^n)^2} \\ (\triangle p_t^\varepsilon, p_t^\varepsilon) &\in \partial L_t(x_{t-1}^\varepsilon, \triangle x_t^\varepsilon) + \frac{\varepsilon}{\lambda}\mathbb{B}_{(\mathbb{R}^n)^2}, \forall t = 1, \cdots, T. \end{aligned}$$

Hence the desired result. □

The following corollary deals with the discrete problem $(\mathrm{P}_{C,F}(l, L))$, that is, the case of Lipschitzian functions l and L_t, explicit set constraint C and set-valued mapping constraint F_t. Before stating the corollary, we need to recall that the graph of the set-valued mapping F_t is the subset

$$\mathrm{gph}\, F_t := \{(u,v) \in \mathbb{R} \times \mathbb{R} \mid v \in F_t(u)\}.$$

In the corollary we assume that the sets C and $\mathrm{gph}\, F_t$ are *closed* in $\mathbb{R}^n \times \mathbb{R}^n$.

Corollary 3.2. *Assume that the functions l and L_t are locally Lipschitzian and bounded from below for all $t = 1, \cdots, T$. Then there exists x_ε such that the qualification condition $\tilde{Q}(x_\varepsilon)$, holds.*

$$\begin{cases} \text{The only vector } y = (y_0, \cdots, y_T) \in (\mathbb{R}^n)^{T+1} \text{ for which} \\ (y_0, -y_T) \in \mathrm{N}_C\left(x_0^\varepsilon, x_T^\varepsilon\right) \text{ et } (\triangle y_t, y_t) \in \mathrm{N}_{\mathrm{gph} F_t}\left(x_{t-1}^\varepsilon, \triangle x_t^\varepsilon\right), \forall t = 1, ..., T, \\ \text{is the zero in } (\mathbb{R}^n)^{T+1}. \end{cases}$$

And there exists some vector $p_\varepsilon = (p_0^\varepsilon, \cdots, p_T^\varepsilon) \in (\mathbb{R}^n)^{T+1}$ such that:

a) $(p_0^\varepsilon, -p_T^\varepsilon) \in \partial l(x_0^\varepsilon, x_T^\varepsilon) + \mathrm{N}_C(x_0^\varepsilon, x_T^\varepsilon) + \varepsilon \mathbb{B}^*_{(\mathbb{R}^n)^2}$.

b) $(\Delta p_t^\varepsilon, p_t^\varepsilon) \in \partial L_t(x_{t-1}^\varepsilon, \Delta x_t^\varepsilon) + \mathrm{N}_{\mathrm{gph}\, F_t}(x_{t-1}^\varepsilon, \Delta x_t^\varepsilon) + \varepsilon \mathbb{B}^*_{(\mathbb{R}^n)^2}$ for all $t = 1, \cdots, T$.

Proof. Put $S_t = \mathrm{gph} F_t$ for all $t = 1, \cdots, T$. Consider the functions

$$\begin{aligned} \tilde{l}_\varepsilon (x_0, x_T) &= l_\varepsilon (x_0, x_T) + \delta_C (x_0, x_T), \\ \tilde{L}_{\varepsilon,t} (x_{t-1}, \triangle x_t) &= L_{\varepsilon,t} (x_{t-1}, \triangle x_t) + \delta_{S_t} (x_{t-1}, \triangle x_t), \end{aligned}$$

which are lsc and proper. Let us prove that the qualification conditionn$Q(x^\varepsilon)$ of Theorem 3.1 holds for the functions $\tilde{l}_\varepsilon$ and $\tilde{L}_{\varepsilon,t}$ for all $t = 1, \cdots, T$. So let $y \in (\mathbb{R}^n)^{T+1}$ such that

$$(y_0, -y_T) \in \partial_\infty \tilde{l}_\varepsilon(x_0^\varepsilon, x_T^\varepsilon),$$

and

$$(\Delta y_t, y_t) \in \partial_\infty \tilde{L}_{\varepsilon,t}(x_{t-1}^\varepsilon, \Delta x_t^\varepsilon), \forall t = 1, \cdots, T.$$

As l_ε, $L_{\varepsilon,t}$ are locally Lipschitzian functions for all $t = 1, \cdots, T$, we see first that by (8) and (9)

$$(y_0, -y_T) \in \partial_\infty \tilde{l}_\varepsilon(x_0^\varepsilon, x_T^\varepsilon) \subset \mathrm{N}_C\left(x_0^\varepsilon, x_T^\varepsilon\right),$$

and also

$$(\Delta y_t, y_t) \in \partial_\infty \tilde{L}_{\varepsilon,t}(x_{t-1}^\varepsilon, \Delta x_t^\varepsilon) \subset \mathrm{N}_{\mathrm{gph} F_t}\left(x_{t-1}^\varepsilon, \triangle x_t^\varepsilon\right), \forall t = 1, \cdots, T.$$

By the qualification condition $\tilde{Q}(x^\varepsilon)$ we have $y_0 = y_1 = \cdots = y_T = 0$, that is, the qualification condition $Q(x^\varepsilon)$ is satisfied.

Since $\tilde{l}_\varepsilon$ and $\tilde{L}_{\varepsilon,t}$ are proper and lsc for all $t = 1, \cdots, T$ and since the qualification condition $Q(x^\varepsilon)$ relative to the problem associated with $\tilde{l}_\varepsilon$ and $\tilde{L}_{\varepsilon,t}$ holds, and by applying the theorem 3.1 we obtain some vector $p_\varepsilon = \left(p_0^\varepsilon, \cdots, p_T^\varepsilon\right) \in (\mathbb{R}^n)^{T+1}$ such that:

a) $\left(p_0^\varepsilon, -p_T^\varepsilon\right) \in \partial l_\varepsilon\left(x_0^\varepsilon, x_T^\varepsilon\right) + \mathrm{N}_C\left(x_0^\varepsilon, x_T^\varepsilon\right)$.

b) $(\triangle p_t^\varepsilon, p_t^\varepsilon) \in \partial L_{t,\varepsilon}\left(x_{t-1}^\varepsilon, \triangle x_t^\varepsilon\right) + \mathrm{N}_{\mathrm{gph} F_t}\left(x_{t-1}^\varepsilon, \triangle x_t^\varepsilon\right), \forall t = 1, \cdots, T.$

Observe also that

$\partial l_\varepsilon\left(x_0^\varepsilon, x_T^\varepsilon\right) \subset \partial l\left(x_0^\varepsilon, x_T^\varepsilon\right) + \varepsilon \mathbb{B}_{(\mathbb{R}^n)^2}$ et $\partial L_{t,\varepsilon}\left(x_{t-1}^\varepsilon, x_t^\varepsilon\right) \subset \partial L_t\left(x_{t-1}^\varepsilon, \triangle x_t^\varepsilon\right) + \varepsilon \mathbb{B}_{(\mathbb{R}^n)^2}$.

Finally we conclude that

a) $\left(p_0^\varepsilon, -p_T^\varepsilon\right) \in \partial l\left(x_0^\varepsilon, x_T^\varepsilon\right) + \mathrm{N}_C\left(x_0^\varepsilon, x_T^\varepsilon\right) + \varepsilon \mathbb{B}_{(\mathbb{R}^n)^2}$.

b) $(\triangle p_t^\varepsilon, p_t^\varepsilon) \in \partial L_t\left(x_{t-1}^\varepsilon, \triangle x_t^\varepsilon\right) + \mathrm{N}_{\mathrm{gph} F_t}\left(x_{t-1}^\varepsilon, \triangle x_t^\varepsilon\right) + \varepsilon \mathbb{B}_{(\mathbb{R}^n)^2}, \forall t = 1, \cdots, T.$ □

Let C_0 be a nonempty closed subset of $\mathbb{R}^n$. The next corollary concerns the minimization problem $(\mathcal{P}_{C_0,F}(g,L))$ where the objective is to minimize the function

$$x \mapsto g(x_T) + \sum_{t=1}^{T} L_t(x_t, \Delta x_t),$$

under the initial constraint $x_0 \in C_0$ and the inclusion constraints $\Delta x_t \in F_t(x_{t-1})$ for all $t = 1, \cdots, T$.

Corollary 3.3. *Let $x_\varepsilon \in (\mathbb{R}^n)^{T+1}$ be an approximation solution of problem $(P_{C_0,F}(g,L))$.*

Assume that the functions g and L_t are locally Lipschitzian for all $t = 1, \cdots, T$, and that the following qualification condition $Q(x_\varepsilon)$ holds:

the only vector $y = (y_0, \cdots, y_T) \in (\mathbb{R}^n)^{T+1}$ for which

$y_0 \in N_{C_0}(x_0), y_T = 0$, and $(\Delta y_t, y_t) \in \mathrm{N}_{\mathrm{gph} F_t}(x_{t-1}, \Delta x_t), \forall t = 1, \cdots, T$ is the zero vector in $(\mathbb{R}^n)^{T+1}$.

Then there exists some vector $p = (p_0^\varepsilon, \cdots, p_T^\varepsilon) \in (\mathbb{R}^n)^{T+1}$ such that:

a) $p_0 \in \mathrm{N}_{C_0}(x_0^\varepsilon) + \varepsilon\mathbb{B}^*_{\mathbb{R}^n}, p_T \in -\partial g(x_T^\varepsilon) + \varepsilon\mathbb{B}^*_{\mathbb{R}^n}$,

b) $(\Delta p_t, p_t) \in \partial L_t(x_{t-1}^\varepsilon, \triangle x_t^\varepsilon) + \mathrm{N}_{\mathrm{gph} Ft}(x_{t-1}^\varepsilon, \triangle x_t^\varepsilon) + \varepsilon\mathbb{B}^*_{\mathbb{R}^n}$ for all $t = 1, \cdots, T$.

Proof. Put $l(x_0, x_T) := g(x_T)$ and $C := C_0 \times \mathbb{R}^n$. Then the normal cone to C is given by $N_C(x_0, x_T) = N_{C_0}(x_0) \times \{0\}$ and the function l is obviously locally Lipschitzian with the equality $\partial l(x_0, x_T) = \{0\} \times \partial g(x_T)$. Further, it is easily seen that the qualification condition $\tilde{Q}(x_\varepsilon)$ holds. Thus, the result is a consequence of Corollary (3.2). □

The previous corollary is relative to the case when the images of the set-valued mappings F_t are prox-regular. Recall that a closed subset S of $\mathbb{R}^n$ is *ρ-prox-regular* (for some $\rho \in]0, +\infty]$) when for any point z of the ρ-open tube

$$U_\rho(S) := \{u \in \mathbb{R}^n \mid d(u, S) < \rho\}.$$

(where $d(\cdot, S)$ is the distance to S with respect to the Euclidean norm), the set S has a unique nearest point (denoted by $P_S(z)$) to z. Recall also that for any set-valued mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ the *coderivative* of G at a point $(u, v) \in \mathrm{gph}\, G$ is the set-valued mapping $D^*G(u, v) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ given by $\zeta \in D^*G(u, v)(\xi)$ if and and only if $(\zeta, -\xi) \in N(\mathrm{gph}\, G, (u, v))$.

We will also need the Lipschitz property concept for set-valued mapping. Recall that the set-valued mapping G is locally Lipschitzian around a point $\bar{u}$ with a non-negative number γ for Lipschitz modulus provided that there exists some positive number η such that for all $u, u' \in B(\bar{u}, \eta)$ one has

$$F(u') \subset F(u) + \|u - u'\|\mathbb{B},$$

where $\mathbb{B}$ denotes the closed unit ball of $\mathbb{R}^n$ centered at the origin. We can now state the corollary for the problem $(\mathcal{P}_{C_0,F}(g))$ where each function L_t is equal to the *null* function.

Corollary 3.4. *Assume that the function g is locally Lipschitzian and that each set-valued mapping F_t is locally Lipschitzian. Then there exists x^{ε}_{t-1} for $t = 1, \cdots, T$. And there exists some vector $p_{\varepsilon} = (p^{\varepsilon}_0, \cdots, p^{\varepsilon}_T) \in (\mathbb{R}^n)^{T+1}$ such that:*

a) $p^{\varepsilon}_0 \in \mathrm{N}_{C_0}(x^{\varepsilon}{}_0) + \varepsilon\mathbb{B}^*_{\mathbb{R}^n}$, $p^{\varepsilon}_T \in -\partial g(x^{\varepsilon}{}_T) + \varepsilon\mathbb{B}^*_{\mathbb{R}^n}$,

b) $(\Delta p^{\varepsilon}_t, p^{\varepsilon}_t) \in \mathrm{N}_{\mathrm{gph}\, F_t}(x^{\varepsilon}_{t-1}, \Delta x^{\varepsilon}_t) + \varepsilon\mathbb{B}^*_{(\mathbb{R}^n)^2}$ *for all* $t = 1, \cdots, T$, *that is,*

$$\exists\, (x_1, x_2) \in \mathbb{B}^*_{(\mathbb{R}^n)^2} / \, (\Delta p^{\varepsilon}_t - \varepsilon x_1) \in D^* F_t(x^{\varepsilon}_{t-1}, \Delta x^{\varepsilon}_t)(-p^{\varepsilon}_t + \varepsilon x_2).$$

Proof. From (3.3) with the function L_t is equal to the *null* function we find

a) $p^{\varepsilon}_0 \in \mathrm{N}_{C_0}(x^{\varepsilon}{}_0) + \varepsilon B^*_{\mathbb{R}^n}$, $p^{\varepsilon}_T \in -\partial g(x^{\varepsilon}{}_T) + \varepsilon\mathbb{B}^*_{\mathbb{R}^n}$,

b) $(\Delta p^{\varepsilon}_t, p^{\varepsilon}_t) \in \mathrm{N}_{\mathrm{gph}\, F_t}(x^{\varepsilon}_{t-1}, \Delta x^{\varepsilon}_t) + \varepsilon\mathbb{B}^*_{(\mathbb{R}^n)^2}$ for all $t = 1, \cdots, T$,

it is easy to observe that:

$\exists\, (x_1, x_2) \in \mathbb{B}^*_{(\mathbb{R}^n)^2}$ satisfying $(\Delta p^{\varepsilon}_t - \varepsilon x_1, p^{\varepsilon}_t - \varepsilon x_2) \in \mathrm{N}_{\mathrm{gph}\, F_t}(x^{\varepsilon}_{t-1}, \Delta x^{\varepsilon}_t)$,

then

$$\Delta p^{\varepsilon}_t - \varepsilon x_1 \in D^* F_t(x^{\varepsilon}_{t-1}, \Delta x^{\varepsilon}_t)(-p^{\varepsilon}_t + \varepsilon x_2).$$

□

In the case when the functions l and L_t are convex continuous, we will use the notion of the relative interiors of the domain of the functions l and L_t.

Theorem 3.5. *Assume that the functions l and L_t are convex continuous (non necessarily lsc) for all $t = 1, \cdots, T$ and assume also the qualification condition holds:*

$$\begin{cases} \text{There exists some point } y = (y_0, \cdots, y_T) \in (\mathbb{R}^n)^{T+1} \text{ such that} \\ (y_0, -y_T) \in \mathrm{ri\ dom}l \text{ and } (\Delta y_t, y_t) \in \mathrm{ri\ dom}L_t, \forall t = 1, ..., T. \end{cases}$$

Then, there exists an approximate solution $x_{\varepsilon} \in (\mathbb{R}^n)^{T+1}$ of the problem $(\mathcal{P}(l, L))$ if and only if there exists some vector $p_{\varepsilon} \in (\mathbb{R}^n)^{T+1}$ satisfying relations (a) and (b) of Theorem 3.1.

Proof. The proof of this theorem is a direct application of the two theorems 3.1 and the Theorem 3.7 [31]. □

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